Tutorial 1

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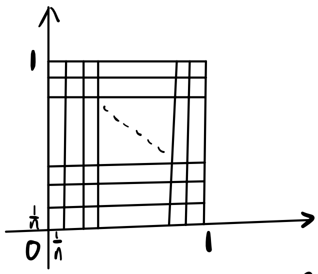
2nd week

1. Use the definition of double integral to evaluate the following double integral

$$\int_0^1 \int_0^1 xy \, \mathrm{d}x\mathrm{d}y$$

Consider the integral region, a rectangle $R: 0 \le x \le 1,\ 0 \le y \le 1$



For any $n > 0$, we separate $0 \le x \le 1$ and $0 \le y \le 1$ into n segments with length $\frac{1}{n}$. Then we get n^2 small rectangles with width $\frac{1}{n}$ and height $\frac{1}{n}$. Let $x_k = \frac{k}{n}$, $y_k = \frac{k}{n}$, $0 \le k \le n-1$. (x_i, y_j) is the left bottom of each small rectangle. The area of each small rectangle is $\frac{1}{n^2}$

Now we compute the Riemann Sum

$$S_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(x_i, y_j) \cdot \frac{1}{n^2}$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \frac{i}{n} \cdot \frac{j}{n} \cdot \frac{1}{n^2} = \frac{1}{n^4} \sum_{i=0}^{n-1} \left(i \sum_{j=0}^{n-1} j\right)$$

$$= \frac{1}{n^4} \sum_{i=0}^{n-1} \left(i \cdot \frac{n(n-1)}{2}\right) = \frac{n-1}{2n^3} \sum_{i=0}^{n-1} i$$

$$= \frac{n-1}{2n^3} \cdot \frac{n(n-1)}{2} = \frac{(n-1)^2}{4n^2}$$

Then we could compute the integral

$$\int_0^1 \int_0^1 xy \, dx\, dy = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{(n-1)^2}{4n^2} = \frac{1}{4}$$

2. Evaluate the iterated integral

$$\int_0^1 \int_0^1 \frac{y}{1+xy} \mathrm{d}x\mathrm{d}y$$

Let $A(y) = \int_0^1 \frac{y}{1+xy} dx$. Then

$$A(y) = \int_0^1 \frac{y}{1+xy} dx = \ln(1+xy)\Big|_{x=0}^{x=1} = \ln(1+y)$$

Hence

$$\int_0^1 \int_0^1 \frac{y}{1+xy} dxdy = \int_0^1 A(y)dy = \int_0^1 \ln(1+y)\,dy$$

$$= y\ln(1+y)\Big|_0^1 - \int_0^1 y\, d\ln(1+y)$$

$$= \ln 2 - \int_0^1 \frac{y}{1+y} dy$$

$$= \ln 2 - \int_0^1 1 - \frac{1}{1+y} dy$$

$$= \ln 2 - \left[y - \ln(1+y)\right]_0^1$$

$$= \ln 2 - [1 - \ln 2] = 2\ln 2 - 1$$

3. Evaluate the double integral over the given region R.

$$\iint_R y\sin(x+y)dA, \ R: \ -\pi \leqslant x \leqslant 0, \ 0 \leqslant y \leqslant \pi$$

Let $A(y) = \int_{-\pi}^0 y \sin(x+y)\, dx$

$$A(y) = -y\cos(x+y)\Big|_{x=-\pi}^{x=0}$$

$$= -y\cos(y) + y\cos(-\pi+y)$$

$$= -y\cos(y) - y\cos(y)$$

$$= -2y\cos y$$

Then

$$\iint_R y\sin(x+y)dA = \int_0^\pi A(y)dy = \int_0^\pi -2y\cos y\, dy$$

$$= -2\int_0^\pi y\, d\sin y$$

$$= -2\left[y\sin y\Big|_0^\pi - \int_0^\pi \sin y\, dy\right]$$

$$= -2\left[\cos y\Big|_0^\pi\right] = -2[-1-1] = 4$$

4. Find the volume of the region bounded above by the paraboloid $z = x^2+y^2$ and below by the square $R: -1 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 1$.

The volumn

$$V = \int_{-1}^{1}\int_{-1}^{1} x^2+y^2 dxdy$$

Let $A(y) = \int_{-1}^{1} x^2+y^2 dx$

$$A(y) = \int_{-1}^{1} x^2+y^2 dx = \left[\tfrac{1}{3}x^3+y^2x\right]\Big|_{-1}^{1} = \left[\tfrac{1}{3}+y^2\right] - \left[-\tfrac{1}{3}-y^2\right] = \tfrac{2}{3}+2y^2$$

Hence $V = \int_{-1}^{1} A(y)dy = \int_{-1}^{1} \frac{2}{3}+2y^2 dy = \left(\frac{2}{3}y+\frac{2}{3}y^3\right)\Big|_{-1}^{1}$

$$= \left(\tfrac{2}{3}+\tfrac{2}{3}\right) - \left(-\tfrac{2}{3}-\tfrac{2}{3}\right)$$

$$= \frac{8}{3}$$

5. Use Fubini's Theorem to evaluate

$$\int_0^2\int_0^1 \frac{x}{1+xy}\mathrm{d}x\mathrm{d}y$$

By Fubini's theorem $\int_0^2\int_0^1 \frac{x}{1+xy}dxdy = \int_0^1\int_0^2 \frac{x}{1+xy}dy\,dx$

Let $A(x) = \int_0^2 \frac{x}{1+xy}dy$

$$A(x) = \ln(1+xy)\Big|_{y=0}^{y=2} = \ln(1+2x)$$

Hence $\int_0^2\int_0^1 \frac{x}{1+xy}dx\,dy = \int_0^1\int_0^2 \frac{x}{1+xy}dy\,dx$

$$= \int_0^1 A(x)dx = \int_0^1 \ln(1+2x)dx$$

$$= x\ln(1+2x)\Big|_0^1 - \int_0^1 x\, d\ln(1+2x)$$

$$= \ln 3 - \int_0^1 \frac{2x}{1+2x}dx$$

$$= \ln 3 - \int_0^1 1-\frac{1}{1+2x}dx$$

$$= \ln 3 - \left[x-\tfrac{1}{2}\ln(1+2x)\right]\Big|_0^1$$

$$= \ln 3 - \left[1-\tfrac{1}{2}\ln 3\right]$$

$$= \frac{3}{2}\ln 3 - 1$$